THE FIRST NON-VANISHING QUADRATIC TWIST OF AN AUTOMORPHIC L-SERIES

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ABSTRACT. Let π be an automorphic representation on $\mathrm{GL}(r, \mathbb{A}_{\mathbb{Q}})$ for $r = 1, 2$, or 3. Let d be a fundamental discriminant and χ_d the corresponding quadratic Dirichlet character. We consider the question of the least d, relative to the data (level, weight or eigenvalue) of π, such that the central value of the twisted L-series is nonzero, i.e. $L(1/2, \pi \otimes \chi_d) \neq 0$.

For example, let N be the level of π. Using multiple Dirichlet series, we prove the nonvanishing of a central twisted L-value with $|d| \ll_{\varepsilon} N^{1/2+\varepsilon}$ for GL(1), $|d| \ll_{\varepsilon} N^{1+\varepsilon}$ for GL(2), and $|d| \ll_{\varepsilon} N^{2+\varepsilon}$ for GL(3), the last case assuming that a certain character is quadratic (see Theorem 1.13). We work over $\mathbb{Q}$ for simplicity but the method generalizes to arbitrary number fields.

We conjecture that in all cases there should be such a twist with $|d| \ll_{\varepsilon} N^{\varepsilon}$. This would follow from a Lindelöf-type bound for a multiple Dirichlet series which does not have an Euler product, but is constructed from a Rankin-Selberg integral applied to automorphic forms which are eigenfunctions of Hecke operators.

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1. Introduction

Much work has been devoted to the problem of bounding the least prime in an arithmetic progression. The Grand Riemann Hypothesis predicts that for $(a, q) = 1$ and any $\varepsilon > 0$,

$$\sum_{\substack{p \equiv a(q) \\ p < N}} \log p = \frac{N}{\varphi(q)} + O_\varepsilon\left(N^{1/2+\varepsilon}\right), \qquad \text{as} \qquad N \to \infty. \qquad (1.1)$$

As $\varphi(q) = q^{1+o(1)}$, the above implies that the main term dominates the error as soon as $N \gg_\varepsilon q^{2+\varepsilon}$. Hence the left hand side of (1.1) is non-zero, confirming the existence of a prime

$$p \ll_\varepsilon q^{2+\varepsilon} \qquad (1.2)$$

with $p \equiv a(q)$.

Unconditionally, the error term in (1.1) is not much better than a power of log savings, the consequence being an exponential rather than polynomial bound in (1.2). Linnik [Lin44] was the first to show that the problem of least prime in an arithemetic progression is not subordinate to progress towards the "first moment" in (1.1), proving a polynomial bound (the quality of which has since been vastly improved).

In this paper we consider the following similar problem. Let π be an automorphic representation on $\mathrm{GL}(r, \mathbb{A}_\mathbb{Q})$, $r = 1, 2$ or 3, and let $\mathcal{D}$ denote the set of fundamental discriminants. For $d \in \mathcal{D}$, let χ_d be the corresponding quadratic Dirichlet character of modulus d. A great deal of attention has been paid in recent years to the question of the existence and abundance of $d \in \mathcal{D}$ such that the central value of the standard L-function attached to π twisted by χ_d, $L(1/2, \pi \otimes \chi_d)$, does not vanish. We pose the following more refined problem.

Question 1.3. If such a d exists, then what is the least value of $|d|$, relative to the data of π (such as its level N or eigenvalue λ), for which the twisted L-series does not vanish at the center?

Remark 1.4. Though one might *a priori* assume that the "analytic conductor" of Iwaniec-Sarnak is a suitable measure of the "complexity" of π in this problem, the following examples suggest in fact that the dependence on the archimedean place plays a substantially different role from the finite ramification.

(1) For π a discrete series representation of $\mathrm{GL}(2)$ corresponding to a holomorphic modular form of level N, by simply combining Waldspurger's theorem [Wal81] with Riemann-Roch, one can

show the existence of a $|d| \ll_\varepsilon N^{1+\varepsilon}$ such that the central L-value is non-zero.

(2) Similarly, for π a tempered representation of GL(2) having Casimir eigenvalue λ,[1] one can apply Waldspurger's theorem and arguments dating back to Maass[2] to show the existence of a nonvanishing central twist χ_d with $|d| \ll \lambda^{1/2}$.

As the analytic conductor in the above examples is roughly $N\lambda$, it is clear that one should separate the level and eigenvalue aspects in this problem. That said, see the caveat in Remark 1.21.

The examples above can be considered "convexity" bounds towards Question 1.3, for reasons which shall become clear, see §1.3. In this paper, we demonstrate the convexity bound for the central L-value of π on GL(r) with $r = 1, 2$, and 3.

1.1. Statements of the Main Results.

We shall really only work with the standard L-function attached to quadratic twists of π, and not π itself. To this end, we make the following

Definition 1.5. By an automorphic L-series $L(s, \pi)$ on GL(r), we mean the following. Assume that the series

$$L(s, \pi) = \sum_{n \geq 1}^{\infty} \frac{c(n)}{n^s}$$

converges absolutely for $\mathfrak{Re}(s)$ sufficiently large, has Euler product

$$L(s, \pi) = \prod_p \prod_{j=1}^{r} \left(1 - \alpha_p^{(j)} p^{-s}\right)^{-1}, \qquad (1.6)$$

and analytic continuation with functional equation

$$\Lambda(s, \pi) := N^{s/2} G_\pi(s) L(s, \pi) = \epsilon_\pi \Lambda(1 - s, \tilde{\pi}). \qquad (1.7)$$

Here $\tilde{\pi}$ is the contragradient of π, $|\epsilon_\pi| = 1$ is the root number, the integer $N \geq 1$ is the level, and

$$G_\pi(s) = \pi^{-rs/2} \prod_{j=1}^{r} \Gamma\left(\frac{s + \kappa_j}{2}\right) \qquad (1.8)$$

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[1]Classically, this corresponds to either a Maass form of eigenvalue $\lambda = \frac{1}{4} + t^2$, $t > 0$, or a holomorphic form of even weight k with $\lambda = \frac{k}{2}(1 - \frac{k}{2})$.

[2]P. Sarnak pointed us to J. Huntley's thesis, where these ideas are vastly generalized.

is a product of archimedean gamma factors with $\mathfrak{Re}(\kappa_j) \geq 0$. We define the "archimedean conductor" of π to be:

$$\mathfrak{q} := \prod_{j=1}^{r} \left(3 + |\kappa_j|\right). \tag{1.9}$$

For positive square-free d, the twisted L-series has Euler product

$$L(s, \pi \otimes \chi_d) = \sum_n \frac{c(n)\chi_d(n)}{n^s} = \prod_p \prod_{j=1}^{r} \left(1 - \chi_d(p)\alpha_p^{(j)} p^{-s}\right)^{-1}, \tag{1.10}$$

and its functional equation is given by

$$\begin{aligned} \Lambda(s, \pi \otimes \chi_d) &:= (\mathfrak{c}(\pi \otimes \chi_d))^{s/2} G_{d,\pi}(s) L(s, \pi \otimes \chi_d) \tag{1.11} \\ &= \epsilon_\pi \psi(d) \Lambda(1 - s, \tilde{\pi} \otimes \chi_d). \end{aligned}$$

Here $G_{d,\pi}$ is a product of gamma factors depending only on π and the sign of d, ψ is a character modulo N, and $\mathfrak{c}(\pi \otimes \chi_d)$ is the conductor of the twisted automorphic representation $\pi \otimes \chi_d$. For example, if $(N, d) = 1$ then

$$\mathfrak{c}(\pi \otimes \chi_d) = N|D|^r, \tag{1.12}$$

where $D = 4d$ or $D = d$ is the conductor of χ_d. For the reader's convenience, we record the possible values of $\mathfrak{c}(\pi \otimes \chi_d)$ in the case $(N, d) \neq 1$ in §2.2.

We first state our main result in the level aspect:

Theorem 1.13. *Let $L(s, \pi)$ be an automorphic L-series on $\mathrm{GL}(r)$ of level N and degree $r = 1, 2$ or 3. Suppose that the root number of $L(s, \pi)$ is not equal to -1 (and hence there exists a non-vanishing quadratic twist, see §3.4). Suppose also in the case $r = 3$ that ψ is trivial or quadratic.*[3] *Then*

- *for $r = 1$, there exists some $|d| \ll_\varepsilon N^{1/2+\varepsilon}$,*
- *for $r = 2$, there exists some $|d| \ll_\varepsilon N^{1+\varepsilon}$, and*
- *for $r = 3$, there exists some $|d| \ll_\varepsilon N^{2+\varepsilon}$,*

such that $L(1/2, \pi \otimes \chi_d) \neq 0$.

In the eigenvalue aspect (which only makes sense over $\mathbb{Q}$ for degree $r = 2$ and $r = 3$), we have:

Theorem 1.14. *Let $L(s, \pi)$ be an automorphic L-series on $\mathrm{GL}(r)$ of archimedean conductor $\mathfrak{q}$ as in (1.9) and degree $r = 2$ or 3. Suppose that the root number of $L(s, \pi)$ is not equal to -1. Then*

- *for $r = 2$, there exists some $|d| \ll_\varepsilon \mathfrak{q}^{1/2+\varepsilon}$, and*
- *for $r = 3$, there exists some $|d| \ll_\varepsilon \mathfrak{q}^{1+\varepsilon}$,*

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[3]This is but a simplifying assumption; for the most general statement, see §4.

such that $L(1/2, \pi \otimes \chi_d) \neq 0$.

Remark 1.15. The archimedean conductor $\mathfrak{q}$ should not be confused with the Casimir eigenvalues of π. Consider a principal series representation π on GL(3) corresponding to a Maass form of type $\nu = (1/3 + it_1, 1/3 + it_2)$, with $t_j \asymp T$. The eigenvalue of the Laplacian is then

$$\lambda = 1 + 3t_1^2 + 3t_1 t_2 + 3t_2^2 \asymp T^2.$$

The Gamma factors of π are (cf. [Gol06, Theorem 6.5.15])

$$G_\pi(s) := \Gamma\left(\frac{s + 2it_1 + it_2}{2}\right) \Gamma\left(\frac{s + it_1 - it_2}{2}\right) \Gamma\left(\frac{s - it_1 - 2it_2}{2}\right),$$

so generically the archimedean conductor is

$$\mathfrak{q} \asymp T^3 \asymp \lambda^{2/3}.$$

Then Theorem 1.14 exhibits a nonvanishing twist χ_d with

$$|d| \ll_\varepsilon \lambda^{3/2 + \varepsilon}.$$

On the other hand, on GL(2), $\mathfrak{q} \asymp \lambda$.

1.2. **Outline of the Proof.**

1.2.1. *The "Moment" Method.*

Let $L(s, \pi)$ be an automorphic L-series on GL(r), $r = 1, 2$, or 3. As in the case of primes in progressions, one can try to compute the first moment:

$$\sum_{\substack{d \in \mathcal{D} \\ |d| < X}} L(1/2, \pi \otimes \chi_d) = M_\pi(X) + E_\pi(X). \tag{1.16}$$

For X large enough that the main term dominates the error, the above formula will produce a non-vanishing central twist. In practice,

$$M_\pi(X) = X^{1 + o(1)},$$

and in order to prove, say Theorem 1.13 in the level N aspect, one needs to bound the error term $E_\pi(X)$ by terms of the form

$$X^{1 - \alpha} N^{\alpha \cdot \theta},$$

for some $0 < \alpha < 1$ with $\theta = 1/2$, $\theta = 1$, or $\theta = 2$ corresponding to GL(1), GL(2), or GL(3), respectively. Even with smooth weights, unconditional moments with this quality of error seem difficult to achieve with existing methods, especially on higher rank groups such as GL(3) (see Remark 1.24). As in Linnik's problem, we will establish first non-vanishing results *without* making progress towards (1.16).

1.2.2. *The "Multiple Dirichlet Series" Method.*

Instead of computing the moment, we employ the theory of double Dirichlet series. Consider the following Dirichlet series, whose coefficients are themselves twisted L-functions with some carefully chosen weights:

$$Z(s, w) := \sum_d \frac{L(s, \pi \otimes \chi_d) \ P(s, \pi, d)}{d^w}. \qquad (1.17)$$

The series thus defined converges for $\mathfrak{Re}(s), \mathfrak{Re}(w)$ sufficiently large. As has been detailed in many places (e.g. [DGH03, BFH04] etc.), $Z(s, w)$ has meromorphic continuation to all $(s, w) \in \mathbb{C}^2$ with explicitly understood polar divisors, and satisfies a finite group of functional equations, including the transformation

$$(s, w) \mapsto (1 - s, 1 - w). \qquad (1.18)$$

Specializing to $s = 1/2$, one obtains a functional equation of the form

$$G(w)N^{\theta w}Z(1/2, w) \approx \tilde{G}(1 - w)N^{\theta(1-w)}\tilde{Z}(1/2, 1 - w), \qquad (1.19)$$

where G and $\tilde{G}$ are archimedean (Gamma) factors and $\tilde{Z}$ is constructed in a similar way as Z. (The true functional equation is actually a linear combination of terms like $\tilde{Z}$, à la the "scattering matrix" in the functional equation of an Eisenstein series, see e.g. equations (3.21) – (3.23).) Moreover, $Z(1/2, w)$ has a pole at $w = 1$ (and possibly at $w = 3/4$ on GL(3)). By a familiar Tauberian argument resembling an approximate functional equation, we can thus write the residue at $w = 1$ as a finite sum of coefficients of $Z(1/2, w)$ (which are of course the sought-after central L-values), where the length of the sum is the square root of the "conductor" in the functional equation (1.19) for the double Dirichlet series:

$$\text{Residue} \approx \sum_d \frac{L(1/2, \pi \otimes \chi_d) \ P(1/2, \pi, d)}{d^{1/2}} \ V\left(\frac{|d|}{N^\theta}\right) + \text{similar}, \ (1.20)$$

where V is supported in $[1, 2]$, say. Then one immediately arrives at a contradiction if all L-values vanish with $|d| \ll N^{\theta+\varepsilon}$. The same argument applies to the archimedean aspect.[4]

Remark 1.21. In fact one *can* combine the level and eigenvalue Theorems 1.13 and 1.14 into a uniform statement, but this involves knowing the "conductor" for the double Dirichlet series attached to π, and not just π itself. It is in general difficult to predict *a priori* the exact shape

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[4]Also, functional equations of the type (1.19) have been worked out in complete detail over number fields [BFH04], so our approach extends to this setting. For ease of exposition, we will restrict ourselves to $\mathbb{Q}$.

of the functional equation in (1.19) without following through a sequence of functional equations as in §2.1.1 to reach the transformation (1.18). So the appropriate "conductor" cannot elementarily be read off from the functional equation (1.7). See also Remark 2.13.

1.3. **Subconvexity.**

Note that the approximate functional equation (1.20) for the double Dirichlet series is morally equivalent to a "convexity" bound for the series $Z(1/2, w)$ at the central point $w = 1/2$. Indeed, the Tauberian argument alluded to before is to consider essentially

$$\frac{1}{2\pi i} \int_{(2)} G(w + 1/2) Z(1/2, w + 1/2) X^w dw \qquad (1.22)$$

$$= \sum_d \frac{L(1/2, \pi \otimes \chi_d) \, P(1/2, \pi, d)}{d^{1/2}} \, V\left(\frac{|d|}{X}\right).$$

Pull the contour past the pole at $w = 1/2$ all the way to the line $\mathfrak{Re}(w) = -1$, say, after which applying the functional equation and taking $X = N^\theta$ recovers (1.20).

If one were to attempt an improvement via these methods on the exponents in Theorems 1.13 and 1.14, a key ingredient would be a "subconvex" bound for $Z(1/2, w)$ at $w = 1/2$. Any improvement on the convexity bound would lead to a corresponding improvement of these results, and a full Lindelöf-type bound would lead to the existence of a non-vanishing twist with

$$|d| \ll_\varepsilon (\mathfrak{q}N)^\varepsilon.$$

The interesting point is that the L-series $Z(1/2, w)$ does **not** have an Euler product. Of course one does not expect a Lindelöf-type bound to be true in general for L-series without an Euler product. See e.g., [CG06] where a counterexample is constructed. However, it does not seem unreasonable to conjecture a Lindelöf type bound for an L-series without an Euler product when that L-series is constructed from a Rankin-Selberg integral applied to one or more automorphic forms that are themselves eigenfunctions of the relevant Hecke operators. Indeed, we conjecture that the double Dirichlet series $Z(s, w)$ satisfies

$$Z(1/2, 1/2 + it) \ll_\varepsilon (\mathfrak{q}N)^\varepsilon (1 + |t|)^A, \qquad (1.23)$$

for some $A > 0$. Were this to be the case, one could just pull the contour in (1.22) to the line $\mathfrak{Re}(w) = 0$ (still collecting the residue at $w = 1/2$) and estimate away the remaining integral. Since the variable

X is free, one can choose it to be as small as N^{ε}, making the sum on the right hand size of (1.22) have negligible length,[5] cf. Remark 4.2.

Remark 1.24. Unconditionally, one has some polynomial bound in (1.23), see (3.24) – (3.26). One can start with the left hand side of (1.22), except without the Gamma factors, pull the line to $\mathfrak{Re}(w) = 0$, and estimate the error there after extracting the residue. Note that this *does not* recover the same result as Theorems 1.13 and 1.14! In fact, this approach is much closer to that of the "moment" approach described in §1.2.1.

1.4. **Degree** $r \geq 4$.

On $\mathrm{GL}(r)$ with $r \geq 4$, the group of functional equations is no longer a finite Weyl group, but is an infinite Coxeter group, see Remark 2.12. Current technology is incapable in this case of obtaining the analytic continuation of $Z(s, w)$ beyond the critical point $(s, w) = (1/2, 1)$, the sole exception being the recent work by Bucur and Diaconu [BD08] in the function field analogue. Moments for quadratic twists of generic π on $\mathrm{GL}(4)$ and higher are also presently unavailable.

In particular, one cannot yet answer the following enticing question. Given two automorphic forms π and π' on $\mathrm{GL}(2)$, each with a positive sign in their functional equation, does there exist a quadratic twist χ_d such that the two twisted L-series simultaneously do not vanish at the center of the critical strip, i.e. $L(1/2, \pi \otimes \chi_d) L(1/2, \pi' \otimes \chi_d) \neq 0$? Similarly, one cannot yet obtain the second moment of an automorphic form π on $\mathrm{GL}(2)$ twisted by quadratic characters, i.e. an asymptotic formula for

$$\sum_{d < X} L(1/2, \pi \otimes \chi_d)^2, \qquad \text{as } X \to \infty.^6$$

1.5. **Moments of Half-integral Weight Forms.**

By the Shimura correspondence, the questions raised above for $\mathrm{GL}(2)$ are related to questions about twisted moments of half-integral weight forms. Again, the L-series attached to a half-integral weight form $\tilde{f}$ does not have an Euler product, yet it seems likely that if the integral weight Shimura correspondent f is an eigenfunction of the Hecke operators, then $L(s, \tilde{f})$ should satisfy a Lindelöf type bound at the center of

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[5]Note added in print: In recent work, Blomer [Blo09] has succeeded on $\mathrm{GL}(1)$ and level $N = 1$ in proving a subconvex estimate for a double Dirichlet series in the t-aspect. Unfortunately this is the only aspect which does not give applications towards our questions. Of course, it does give evidence that more progress is within reach.

[6] Note added in print: in the recent work [SY09], Soundararajan and Young give unconditional *lower* bounds which match the conjectured main term!

its critical strip. In joint work with Gautam Chinta [CHK10], we have observed that, contrary to the integral weight situation, if one forms the multiple Dirichlet series

$$\tilde{Z}(s_1, s_2, w) \approx \sum_d \frac{L(s_1, \tilde{f} \otimes \chi_d) L(s_2, \tilde{f} \otimes \chi_d)}{d^w},$$

then its group of functional equations is isomorphic to the Weyl group associated to the Dynkin diagram A_5, which is finite! Thus we are able to obtain first and second moments for half-integral weight forms twisted by quadratic characters, i.e. asymptotics as $X \to \infty$ for

$$\sum_{d < X} L(1/2, \tilde{f} \otimes \chi_d) \quad \text{and} \quad \sum_{d < X} L(1/2, \tilde{f} \otimes \chi_d)^2.$$

As the first pole of $\tilde{Z}(1/2, 1/2, w)$ appears at $w = 1$, the second moment is asymptotic to $X\,P(\log X)$, where P is some polynomial. This gives further evidence of the truth of a Lindelöf type bound, even for certain "arithmetic" L-functions without Euler products.

1.6. Simultaneous Non-vanishing Twists.

Choosing the representation π in a particular way, such as

$$\pi = \chi_{N_1} \boxplus \chi_{N_2} \boxplus \chi_{N_3} \qquad \text{or} \qquad \pi = \pi_1 \boxplus \chi_{N_1}$$

for characters χ_{N_j} and π_1 on GL(2), Theorem 1.13 has the following immediate corollary on simultaneously non-vanishing twists.

Corollary 1.25. *Let* $L(s, \chi_{N_1}), L(s, \chi_{N_2}), L(s, \chi_{N_3})$ *be three Dirichlet L-series with conductors* N_1, N_2, N_3. *Let* $L(s, \pi)$ *be an automorphic L-series on* GL(2) *of level* N. *Then*
   (1) *there exists* $|d| \ll (N_1 N_2)^{1+\epsilon}$ *with*

$$L(1/2, \chi_d \chi_{N_1}) L(1/2, \chi_d \chi_{N_2}) \neq 0,$$

   (2) *there exists* $|d| \ll (N_1 N_2 N_3)^{2+\epsilon}$ *such that*

$$L(1/2, \chi_d \chi_{N_1}) L(1/2, \chi_d \chi_{N_2}) L(1/2, \chi_d \chi_{N_3}) \neq 0,$$

   (3) *there exists* $|d| \ll (N_1 N)^{2+\epsilon}$ *such that*

$$L(1/2, \chi_d \chi_{N_1}) L(1/2, \pi \otimes \chi_d) \neq 0.$$

1.7. Outline of the Paper.

In §2, we present the heuristic derivation of the functional equations for the double Dirichlet series $Z(s, w)$, and then state them rigorously in §3. These are well-known to the experts, but our application requires slightly more refined information; the level aspect is the only case which causes difficulty. Equipped with this data, we prove the main theorems in §4.

2. Preliminaries

2.1. **The Heuristic Argument.**

Before presenting the (quite technical) details of the functional equation leading to (1.19), we give a heuristic argument, focusing on the level aspect. It will contain some very imprecise statements regarding functional equations but should nevertheless be a useful reference guide for the actual proofs. We pretend throughout this section, for clarity of exposition, that all numbers are positive and congruent to 1 modulo 4, and that quadratic reciprocity is perfect. We will also suppress the weights $P(s, \pi, d)$; they appear in every equation and contribute little to the exposition.

Let π be an automorphic representation on $\mathrm{GL}(r, \mathbb{A}_{\mathbb{Q}})$, with $r = 1, 2$, or 3 and Fourier coefficients $c(n)$. Consider the following double Dirichlet series:

$$Z(s, w) = \sum_d \frac{L(s, \pi \otimes \chi_d)}{d^w}. \tag{2.1}$$

Very roughly, inserting (1.10) into (2.1), $Z(s, w)$ is represented by the double Dirichlet series

$$\sum_{d,n} \frac{c(n)\chi_d(n)}{d^w n^s}.$$

This suggests that if quadratic reciprocity held perfectly, that is $\chi_d(n) = \chi_n(d)$, then we could rewrite this as

$$Z(s, w) = \sum_n \frac{L(w, \chi_n) c(n)}{n^s}, \tag{2.2}$$

and in fact this interchange can be made rigorous, cf. §3.1.

Applying the functional equation to the numerator of (2.2) and suppressing gamma factors, we see that there is a functional equation sending

$$Z(s, w) \to Z(s + w - 1/2, 1 - w). \tag{2.3}$$

On the other hand, if we apply (1.11) to the numerator of (2.1), we find that there is a functional equation sending

$$Z(s, w) \to N^{1/2-s}\tilde{Z}(1 - s, w + rs - r/2), \tag{2.4}$$

where

$$\tilde{Z}(s,w) = \sum \frac{L(s, \tilde{\pi} \otimes \chi_d)\psi(d)}{d^w} = \sum_{n,d} \frac{\chi_n(d)\psi(d)\tilde{c}(n)}{n^s d^w}. \tag{2.5}$$

Reversing orders of summation and collecting terms in (2.5), we find that

$$\tilde{Z}(s,w) = \sum_n \frac{L(w, \chi_n\psi)\tilde{c}(n)}{n^s}.$$

Remark 2.6. In the above, the conductor of ψ could be any divisor of N. In this heuristic we will assume for simplicity that $\psi^2 = 1$ and that the conductor equals N.

Applying the functional equation to this numerator, we see that there is a transformation

$$\tilde{Z}(s,w) \to N^{1/2-w}\tilde{Z}(s+w-1/2, 1-w). \tag{2.7}$$

Remark 2.8. Note that if ψ is complex, then a new character and a Gauss sum could be introduced in this functional equation. This is why the result for ψ complex is slightly worse than the result for ψ real in the case $r = 3$, cf. (3.29).

Similarly, (2.4) can be used in reverse to give

$$\tilde{Z}(s,w) \to N^{1/2-s}Z(1-s, w+rs-r/2). \tag{2.9}$$

2.1.1. *Iterating Functional Equations.*

We now apply these functional equations in sequence. If the degree $r = 1$, we apply in succession (2.3), (2.4), and (2.7), obtaining

$$\begin{aligned} Z(s,w) &\to Z(s+w-1/2, 1-w) \to N^{1-s-w}\tilde{Z}(3/2-s-w, s) \\ &\to N^{3/2-2s-w}\tilde{Z}(1-w, 1-s). \end{aligned} \tag{2.10}$$

Remark 2.11. On GL(1) there is an extra symmetry in (2.5), namely $\tilde{Z}(s,w) \approx \tilde{Z}(w,s)$, coming from the relation $\psi(d) = c(d)$.

If $r = 2$ we apply in succession (2.3), (2.4), (2.7) and (2.9), obtaining

$$\begin{aligned} Z(s,w) &\to Z(s+w-1/2, 1-w) \to N^{1-s-w}\tilde{Z}(3/2-s-w, w+2s-1) \\ &\to N^{5/2-3s-2w}\tilde{Z}(s, 2-2s-w) \\ &\to N^{3-4s-2w}Z(1-s, 1-w). \end{aligned}$$

If $r = 3$ we apply in succession (2.3), (2.4), (2.7), (2.9), (2.3) and (2.4), obtaining

$$
\begin{aligned}
Z(s, w) \;\; &\rightarrow \;\; Z(s + w - 1/2, 1 - w) \\
&\rightarrow \;\; N^{1-s-w} \tilde{Z}(3/2 - s - w, 3s + 2w - 2) \\
&\rightarrow \;\; N^{7/2-4s-3w} \tilde{Z}(2s + w - 1, 3 - 3s - 2w) \\
&\rightarrow \;\; N^{5-6s-4w} Z(2 - 2s - w, w + 3s - 3/2) \\
&\rightarrow \;\; N^{5-6s-4w} Z(s, 5/2 - 3s - w) \\
&\rightarrow \;\; N^{11/2-7s-4w} \tilde{Z}(1 - s, 1 - w).
\end{aligned}
$$

Remark 2.12. In each of the cases above, finitely-many iterations of the functional equations will return us to $Z(s, w)$. For degree $r \geq 4$, one can cycle the transformations ad infinitum, never arriving at the desired argument $Z(1 - s, 1 - w)$.

When the functional equations above are applied to $Z(1/2, w)$ we find the following relations hold (in each case below, "$\rightarrow$" indicates that only the archimedean contributions are suppressed). When $r = 1$:

$$
N^{w/2} Z(1/2, w) \rightarrow N^{(1-w)/2} \tilde{Z}(1/2, 1 - w),
$$

when $r = 2$:

$$
N^{w} Z(1/2, w) \rightarrow N^{1-w} Z(1/2, 1 - w),
$$

and when $r = 3$:

$$
N^{2w} Z(1/2, w) \rightarrow N^{2(1-w)} \tilde{Z}(1/2, 1 - w).
$$

These are exactly the relations corresponding to (1.19).

Remark 2.13. As noted in Remark 1.21, it is a bit delicate to determine the exact form of the functional equation for $Z(1/2, w)$, and hence its "analytic conductor", given the initial data of π. But one can use the above heuristic as a template to predict the outcome.

2.2. Ramified Conductor.

In this section, we give precise details for the conductor of $\pi \otimes \chi_d$ appearing in (1.11) in the case of ramified twist. We are indebted to Dinakar Ramakrishnan for providing us with the following case by case analysis.

Let π be an irreducible admissible representation of $\mathrm{GL}_r(\mathbb{Q}_p)$ and χ a character of $\mathbb{Q}_p^\times$. (The global conductor is a product of such local conductors, all but finitely many of which are unity.) Recall that $\mathfrak{c}$ denotes the conductor of a representation. The case when the conductors of π and χ are relatively prime is trivial, and one has $\mathfrak{c}(\pi \otimes \chi) = \mathfrak{c}(\pi)\mathfrak{c}(\chi)^r$, as in (1.12).

Let us assume from now on that $\mathfrak{c}(\pi) = p^a$ and $\mathfrak{c}(\chi) = p^b$. The representation $\pi \otimes \chi$ is hence ramified, and one can appeal to Tadic's classification [Tad86] of such. There is a partition $r = r_1 + r_2 + ... + r_m$, and discrete series representations π_j, $1 \le j \le m$, of $\mathrm{GL}_{r_j}(\mathbb{Q}_p)$ such that π is parabolically induced from the representation $\pi_1 \times \pi_2 \times ... \times \pi_m$ of the parabolic P attached to the partition. One has

$$\mathfrak{c}(\pi) = \prod_{j \le m} \mathfrak{c}(\pi_j)$$

and

$$\mathfrak{c}(\pi \otimes \chi) = \prod_{j \le m} \mathfrak{c}(\pi_j \otimes \chi).$$

Thus it suffices to understand $\mathfrak{c}(\eta \otimes \chi)$ for a discrete series representation η of $\mathrm{GL}_t(\mathbb{Q}_p)$ and a character χ. For simplicity, assume that $t < p$.

Suppose $t = 1$. Then η is a character and so $\mathfrak{c}(\eta\chi) \le \max(\mathfrak{c}(\eta), \mathfrak{c}(\chi))$. Consequently, if π is a principal series representation of $\mathrm{GL}_r(\mathbb{Q}_p)$ attached to the characters $\mu_1, ..., \mu_r$, we have

$$\mathfrak{c}(\pi \otimes \chi) = \prod_j \mathfrak{c}(\mu_j \chi) \le \prod_j \max(\mathfrak{c}(\mu_j), \mathfrak{c}(\chi)).$$

Now take $t = 2$ or $t = 3$. As η is a discrete series it is either Steinberg St, or a twisted Steinberg $St(\nu)$ for a character ν of $\mathbb{Q}_p$, or a supercuspidal representation. We then have the following situations:

- If $\eta = St$ then $\mathfrak{c}(\eta) = p$ and $\mathfrak{c}(\eta \otimes \chi) = \mathfrak{c}(\chi) = p^b$.
- If $\eta = St(\nu)$ then $\mathfrak{c}(\eta) = \mathfrak{c}(\nu)$ and $\mathfrak{c}(\eta \otimes \chi)\mathfrak{c}(St(\nu\chi)) = \mathfrak{c}(\nu\chi)$ in the case $\nu\chi \ne 1$, while $\mathfrak{c}(\eta \otimes \chi)\mathfrak{c}(St(\nu\chi)) = p$ if $\nu\chi = 1$.
- If η is supercuspidal (recall we assumed $t < p$), then η is attached to a character λ of a cyclic t-extension K of $\mathbb{Q}_p$. We have $\mathfrak{c}(\eta) = N(\mathfrak{c}(\lambda))d_K$, where N denotes the norm from K to $\mathbb{Q}_p$, and d_K denotes the discriminant of $K/\mathbb{Q}_p$. Moreover, $\eta \otimes \chi$ is attached to the character $\lambda(\chi \circ N)$ of K, so that

  $$\mathfrak{c}(\eta \otimes \chi) = N(\mathfrak{c}(\lambda(\chi \circ N)))d_K \le N\max(\mathfrak{c}(\lambda), \mathfrak{c}(\chi \circ N))d_K.$$

  There are really two types of supercuspidals η, depending on whether $K/\mathbb{Q}_p$ is unramified or ramified. In the former case, $d_K = 1$ and p defines a uniformizer of K, so that (in this case) $\mathfrak{c}(\eta) = \mathfrak{c}(\lambda)$ and $\mathfrak{c}(\eta \otimes \chi) = \max(\mathfrak{c}(\lambda), p^{k/t})$ if $\mathfrak{c}(\chi) = p^k$. Next suppose $K/\mathbb{Q}_p$ is ramified with $d_K = p^x$. In this case, if ϖ is a uniformizer of K, we have $p\mathcal{O}_K = \varpi^t \mathcal{O}_K$ and $N(\varpi) = p$. Thus, if $\mathfrak{c}(\lambda) = \varpi^j$, then $\mathfrak{c}(\eta) = p^{j+x}$, and $\mathfrak{c}(\eta \otimes \chi) = p^{x+\max(j,b)}$.

This completes our analysis.

3. The Functional Equations and Their Properties

3.1. The Interchange Property.

The interchange property alluded to in the transfer from (2.1) to (2.2) is a very well-developed component of the theory of Multiple Dirichlet Series. The exact "correction" polynomials and their properties at unramified places are detailed in many places, including [GH85, BFH96, DGH03, BFH04, CFH06, BBC$^+$06], to name a few. We assume some familiarity with these sources, while making the observation discussed below, that the correction polynomials can also be defined at ramified places.

3.1.1. *The Standard Approach.*

First we recall the standard approach to the interchange property. Let S be a finite set of primes consisting of 2 and the primes dividing the level N of π. Let $M := \prod_{p \in S} p$, and let

$$L^S(s, \pi \otimes \chi) := \prod_{p \notin S} L_p(s, \pi \otimes \chi)$$

denote the twisted L-series with the places dividing M removed. Let $\ell_1, \ell_2 \mid M$ with $\ell_1, \ell_2 > 0$ and $a_1, a_2 \in \{-1, 1\}$. One defines

$$Z^S(s, w; \chi_{a_2\ell_2}, \chi_{a_1\ell_1}; \pi) := \sum_{(d,M)=1} \frac{L^S(s, \pi \otimes \chi_{d_0}\chi_{a_1\ell_1})\chi_{a_2\ell_2}(d_0)P^{(a_1\ell_1)}_{d_0,d_1}(s)}{d^w},$$

(3.1)

where the sum is over $d > 0$ and we use the decomposition $d = d_0 d_1^2$ with d_0 square-free. This allows further character twists inside the L-function by $\chi_{a_1\ell_1}$ and in the numerator by $\chi_{a_2\ell_2}$. One only considers such sums in which all the L-series of the numerator share a common gamma factor. (A given sum can be, if necessary, subdivided into several such sums.)

The "correction" polynomials $P(s)$ appearing in (3.1) are finite Dirichlet series that are uniquely determined by certain functional equations and limiting values. They are introduced to, in essence, extend the level of a character from a square-free d_0 to $d_0 d_1^2$ while retaining a similar functional equation in which the level d_0 is replaced by $d_0 d_1^2$. Note this is *not* the case if the primitive χ of conductor d_0 is simply replaced by the imprimitive character of conductor $d_0 d_1^2$.

Take, for example, the simplest GL(1) case, in which $L_p(s, \pi \otimes \chi)$ is the p-part of a quadratic Dirichlet L-series. Here if $d_0 = q$ and $d_1 = p$ are distinct primes congruent to 1 modulo 4, and $d = qp^2$, then

$$P_{q,p}^{(1)}(s) = 1 - \chi_q(p)p^{-s} + p^{1-2s}.$$

Note that this has a functional equation $P_{q,p}^{(1)}(1-s) = p^{2s-1}P_{q,p}^{(1)}(s)$ and that by the original functional equation for $L(s, \chi_q)$, the product satisfies the particularly nice functional equation

$$(qp^2/\pi)^{s/2}\Gamma(s/2)L(s, \chi_q)P_{q,p}^{(1)}(s) = L^*(s, \chi_q) = L^*(1-s, \chi_q).$$

The now standard fact is that the polynomials $P_{d_0,d_1}^{(a_1\ell_1)}(s)$ can be defined in such a way that the interchange (2.2) can be accomplished, with a new set of correction polynomials $Q(w)$ on the other side.

Proposition 3.2. *There exists a choice of the polynomials $P(s)$ and $Q(w)$ such that*

$$Z^S(s, w; \chi_{a_2\ell_2}, \chi_{a_1\ell_1}; \pi) = \sum_{(n,M)=1} \frac{L^S(w, \tilde{\chi}_{n_0}\chi_{a_2\ell_2})\chi_{a_1\ell_1}(n_0)c(n_0 n_1^2)Q_{n_0,n_1}^{(a_2\ell_2)}(w)}{n^s}.$$

$$(3.3)$$

Here the sum is over $n = n_0 n_1^2$ where $n_0 > 0$ is squarefree, and $\tilde{\chi}_{n_0}$ denotes the quadratic character with conductor n_0 defined by $\tilde{\chi}_{n_0}(*) = \left(\frac{*}{n_0}\right)$. (Recall $2|M$ so $(2, n_0) = 1$.)

The polynomials $Q_{n_0,n_1}^{(a_2\ell_2)}(w)$ have functional equation properties similar to those of $P_{d_0,d_1}^{(a_1\ell_1)}(s)$.

These correction polynomials have been explicitly worked out and written down for every case considered here. The important point is that they exist and they are unique, see e.g. (2.2) and (2.3) of [BFH04]. Their exact form is cumbersome and not particularly illuminating for our purposes, though their combinatorial properties have fascinating connections to statistical mechanics, crystal bases, ice models, etc., cf. [BBF10]. The main points to bear in mind are that

(1) For any d_0, n_0, $P_{d_0,1}^{(a_1\ell_1)}(s) = Q_{n_0,1}^{(a_2\ell_2)}(w) = 1$. That is, the correction polynomials are trivial when coefficients have square free indices.
(2) The P and Q polynomials have simple functional equations that are compatible with the L-series by which they are multiplied.
(3) They permit interchanges such as that transforming (3.1) into (3.3).

(4) The sizes of the P and Q polynomials are sufficiently small that for fixed d_0, n_0 the sums

$$\sum_{d_1} \frac{P_{d_0,d_1}^{(a_1\ell_1)}(s)}{d_1^{2s}} \quad \text{and} \quad \sum_{n_1} \frac{Q_{n_0,n_1}^{(a_2\ell_2)}(w)}{n_1^{2w}}$$

converge absolutely for $\mathfrak{Re}\, s > 1/2$ and $\mathfrak{Re}\, w > 1/2$.

3.1.2. *Ramified Correction Polynomials.*

We now make explicit the aforementioned observation that it is not necessary to remove ramified L-parts to determine polynomials P and Q that satisfy the above four properties. Take, for example the case of GL(1). Here $\pi = \chi$, a character of level N. Taking, for simplicity, $a_1 = \ell_1 = a_2 = \ell_2 = 1$, we replace (3.1) by

$$Z(s, w; 1, 1; \pi) := \sum_{d \geq 1} \frac{L(s, \chi\chi_{d_0})P_{d_0,d_1}^{(1)}(s)}{d^w}. \tag{3.4}$$

Here $L(s, \chi\chi_{d_0})$ denotes the primitive L-series attached to the character $\chi\chi_{d_0}$. Thus if $(d_0, N) = 1$ the conductor of the L-series is Nd_0. In this case $P_{d_0,d_1}^{(1)}(s)$ retains its standard definition, even when d_1 is divisible by primes dividing N. If $(d_0, N) \neq 1$, then one writes $Nd_0 = N_0d_3d_4^2$, where N_0 is the conductor of the product $\chi\chi_{d_0}$ and d_3 is square free. In this case the numerator of the $d_0d_1^2$ term is $L(s, \chi\chi_{d_0})P_{d_3,d_4}^{(1)}(s)$. For example, if $N = p$ is prime and $\chi = \chi_p$ is the quadratic character modulo p, then for any d_1, and $d_0 = pd_0'$, the numerator becomes $L(s, \chi_{d_0'})P_{d_0',pd_1}^{(1)}(s)$. The numerator has been constructed to have the correct functional equation, and all that needs to be verified is the interchange property, which is easily done. In particular, in place of (3.3) we obtain

$$Z(s, w; 1, 1; \pi) = \sum_{n \geq 1} \frac{L(w, \chi_{n_0})\chi(n)Q_{n_0,n_1}^{(1)}(w)}{n^s}. \tag{3.5}$$

The reason why this interchange still holds, even when divisibility at ramified places is not restricted, is that the P and Q polynomials exist because of a uniqueness principle that holds in GL(r) when $r = 1, 2, 3$. See [CFH06, §3.5] for an exposition of this principle and a description of how the properties above are used to determine the P and Q polynomials. In GL(2) the p-part of an L-series corresponding to a ramified prime can have one or zero Satake parameters. If zero, the same discussion as in GL(1) above can be used to determine the p-parts of P and Q polynomials. If one, then the the p-part is identical to

the corresponding parts of the polynomials in the GL(1) context, with the parameter $\alpha = \pm 1$ replacing $\chi(p)$. Similarly, in GL(3), at ramified primes the p-parts of the P, Q polynomials reduce to their GL(1) and GL(2) counterparts corresponding to $0, 1$ or 2 Euler factors.

In conclusion, we have the following (cf. Proposition 3.2).

Proposition 3.6. *For $\ell_1, \ell_2 \in \{1, 2\}$ and $a_1, a_2 \in \{\pm 1\}$, let*

$$Z(s, w; \chi_{a_2\ell_2}, \chi_{a_1\ell_1}; \pi) := \sum_{d \geq 1} \frac{L(s, \pi \otimes \chi_{d_0}\chi_{a_1\ell_1})\chi_{a_2\ell_2}(d_0)P_{d_0,d_1}^{(a_1\ell_1)}(s)}{d^w},$$

where the P polynomial is as described above. Then there is a Q polynomial satisfying the above properties so that

$$Z(s, w; \chi_{a_2\ell_2}, \chi_{a_1\ell_1}; \pi) = \sum_{n \geq 1} \frac{L(w, \tilde{\chi}_{n_0}\chi_{a_2\ell_2})\chi_{a_1\ell_1}(n_0)c(n_0 n_1^2)Q_{n_0,n_1}^{(a_2\ell_2)}(w)}{n^s}.$$

As the presence of the correction polynomials is distracting, we fix the notation

$$L(s, \pi \otimes \chi_d \chi_{a_1\ell_1}) := L(s, \pi \otimes \chi_{d_0}\chi_{a_1\ell_1})P_{d_0,d_1}^{(a_1\ell_1)}(s) \qquad (3.7)$$

and

$$L(w, \tilde{\chi}_n \chi_{a_2\ell_2}) := L(w, \tilde{\chi}_{n_0}\chi_{a_2\ell_2})Q_{n_0,n_1}^{(a_2\ell_2)}(w). \qquad (3.8)$$

Thus the interchange in the order of summation above takes the slightly more reasonable form:

$$
\begin{aligned}
Z(s, w; \chi_{a_2\ell_2}, \chi_{a_1\ell_1}; \pi) \;=\; & \sum_{d \geq 1} \frac{L(s, \pi \otimes \chi_d \chi_{a_1\ell_1})\chi_{a_2\ell_2}(d)}{d^w} \qquad (3.9) \\
=\; & \sum_{n \geq 1} \frac{L(w, \tilde{\chi}_n \chi_{a_2\ell_2})\chi_{a_1\ell_1}(n)c(n)}{n^s}.
\end{aligned}
$$

3.2. The meromorphic continuation.

Modulo the caveat in the previous subsection about ramified correction polynomials, it is now a standard matter to meromorphically continue the series above. For the reader's convenience, we include a sketch of the argument, keeping careful track of the dependence on the level of each occurring transformation.

Proceeding as in [DGH03], collect the set of quadratic characters appearing above into

$$\mathcal{M} := \{\chi_{a\ell} : a = \pm 1, \ \ell = 1, 2\},$$

and assemble the corresponding double Dirichlet series into the 16-dimensional vector

$$\vec{\mathbf{Z}}(s,w;\pi) = \{Z(s,w;\chi_{a_2\ell_2},\chi_{a_1\ell_1};\pi)\}_{\substack{\chi_{a_1\ell_1}\in\mathcal{M}\\\chi_{a_2\ell_2}\in\mathcal{M}}}.$$

Let α and β denote the involutions $\alpha : (s,w) \mapsto (1-s, w+r(s-1/2))$ and $\beta : (s,w) \mapsto (s+w-1/2, 1-w)$. The double Dirichlet series $\vec{\mathbf{Z}}(s,w;\pi)$ converges absolutely in the tube region where the real parts of s and w exceed 1. Also, by the analytic continuation and functional equation of $L(s, \pi \otimes \chi_d\chi_{a_1\ell_1})$, it converges for all s as long as the real part of w is sufficiently large. Similarly, by the interchange in (3.9), it converges for all w as long as the real part of s is sufficiently large. Let R_1 denote the tube region which is the union of all such s,w.

From (3.9) we see that there are potential polar lines at $s = 1$ and $w = 1$. If the orders of these poles are p_1, p_2 respectively then $(1-s)^{p_1}(1-w)^{p_2}\vec{\mathbf{Z}}(s,w;\pi)$ has a holomorphic continuation to R_1.

We now write $Z(s,w;\chi_{a_2\ell_2},\chi_{a_1\ell_1};\pi)$ in its interchanged form

$$Z(s,w;\chi_{a_2\ell_2}\psi,\chi_{a_1\ell_1};\pi) = \sum_{n\geq 1} \frac{L(w,\tilde{\chi}_n\chi_{a_2\ell_2})\chi_{a_1\ell_1}(n)c(n)}{n^s}.$$

Let $G_{a_2\ell_2,n}(w)$ be the gamma function associated to the Dirichlet L-series in the numerator. This will depend upon $a_2\ell_2$ and the congruence class of n modulo 8. Multiplying and dividing each term by the appropriate $G_{a_2\ell_2}(w)$ and applying the β involution (effectively applying the $r = 1$ instance of (1.11)), we obtain

$$\begin{aligned}Z(s,w;\chi_{a_2\ell_2},\chi_{a_1\ell_1};\pi) &= (\delta_1(a_2\ell_2))^{1/2-w}\sum_{n\geq 1}\frac{G_{a_2\ell_2,n}(1-w)}{G_{a_2\ell_2,n}(w)}\\ &\times \frac{L(1-w,\tilde{\chi}_n\chi_{a_2\ell_2})\chi_{a_1\ell_1}(n)c(n)}{n^{s+w-1/2}}.\end{aligned} \quad (3.10)$$

Here $\delta_1(a_2\ell_2)$ is the power of 2 associated to the discriminant of the corresponding quadratic field. Applying β to R_1 maps R_1 into a new tube region which intersects R_1. Inspecting (3.10) we see that two potential additional polar lines have been added: $w = 0$ and $s+w-1/2 = 1$. Canceling these lines we see that

$$(1-s)^{p_1}(1-w)^{p_2}(w)^{p_2}(s+w-1/2)^{p_1}\vec{\mathbf{Z}}(s,w;\pi)$$

has a holomorphic continuation to R_1.

Thus the original function, after the polar lines are cancelled, is extended to a new function which is holomorphic in the region R_2, defined to be the union $R_1 \cup \beta(R_1)$. Note that no new poles are introduced

by the gamma functions in the numerator as their products with the corresponding L series are analytic, except possibly at 0 and 1.

The catch is that the order of summation can not be changed in (3.10) as the sum over n has a varying gamma ratio coefficient, depending upon the congruence class of n modulo 8. This is easily remedied by replacing the congruence class modulo 8 conditions by linear combinations of sums over all n twisted by characters modulo 8. Thus for each $a_1\ell_1, a_2\ell_2$, the right hand side of (3.10) breaks up into a linear combination summed over a modulo 8, with constant coefficients, of pieces of the form

$$(\delta_1(a_2\ell_2))^{1/2-w}\frac{G_{a_2\ell_2,a}(1-w)}{G_{a_2\ell_2,a}(w)}\sum_{n\geq 1}\frac{L(1-w,\tilde{\chi}_n\chi_{a_2\ell_2})\chi_{a_1'\ell_1'}(n)c(n)}{n^{s+w-1/2}} \qquad (3.11)$$

$$= (\delta_1(a_2\ell_2))^{1/2-w}\frac{G_{a_2\ell_2,a}(1-w)}{G_{a_2\ell_2,a}(w)}Z(s+w-1/2,1-w;\chi_{a_2\ell_2},\chi_{a_1'\ell_1'};\pi).$$

The interchange property can now be applied to each of the above pieces. For a further discussion of this detail, see [DGH03], line (4.23) in the arxiv version, and the discussion just preceding it.

Now let $G_\pi^{(a_1\ell_1)}(s+w-1/2)$ be the gamma factor associated to the L-series in the numerator of the reflected series in (3.11). This will be the common gamma factor for all $d \geq 1$. Multiplying and dividing by $G_\pi^{(a_1\ell_1)}(s+w-1/2)$ and using the known functional equation of the L-series in the s variable (see (1.11)), we effectively apply the involution α, obtaining for each $a_1\ell_1$,

$$Z(s+w-1/2,1-w;\chi_{a_2\ell_2},\chi_{a_1\ell_1};\pi) \qquad (3.12)$$

$$= (N\delta_2(a_1\ell_1))^{1-s-w}\frac{G_\pi^{(a_1\ell_1)}(3/2-s-w)}{G_\pi^{(a_1\ell_1)}(s+w-1/2)}$$

$$\times Z(3/2-s-w,rs+(r-1)w+1-r;\chi_{a_2\ell_2}\psi,\chi_{a_1\ell_1};\pi).$$

Here $\delta_2(a_1\ell_1)$ is again the power of 2 associated to the discriminant of the corresponding quadratic field. Also we must be careful to choose regions for the parameters where the sum is absolutely convergent in both variables. Notice that the character $\chi_{a_2\ell_2}$ has been multiplied by the character ψ introduced by the application of the functional equation.

We now have two other potential polar line at $s+w-1/2=0$ and at $rs+(r-1)w+1-r=1$. Canceling these lines we see that

$$(1-s)^{p_1}(1-w)^{p_2}(w)^{p_2}(s+w-1/2)^{p_1}(3/2-s-w)^{p_1}$$

$$\times (r-rs-(r-1)w)^{p_2}\overrightarrow{\mathbf{Z}}(s,w;\pi)$$

has a holomorphic continuation to R_3, the union $R_2 \cup \alpha(R_2)$. Again no new poles are introduced by the gamma functions in the numerator since the product of this with the L-series in the numerator is analytic (except possibly at 0 and 1).

If $r = 1$, the argument of Z has been transformed from (s, w) to $(3/2 - s - w, s)$. In this case the convex hull of R_3 is all of $\mathbb{C}^2$. A theorem of Böchner, see e.g. [Hör90, Thm 2.5.10], then extends the domain of holomorphy of the function in (3.13) to this convex closure, namely $\mathbb{C}^2$. The original function $\vec{\mathbf{Z}}(s, w; \pi)$ extends to a meromorphic function of s, w in $\mathbb{C}^2$ with poles cancelled by

$$P_1(s, w) = s(1 - s)w(1 - w)(s + w - 1/2)(3/2 - s - w),$$

and with polynomial growth in vertical strips determined, as usual, by the convexity principle. A very detailed explanation of this process in the case $r = 3$, along with illustrations, is given in [DGH03, Proposition 4.11].

In the case $r = 2$ an additional application of the β involution is applied to $Z(3/2 - s - w, w + 2s - 1; \chi_{a_2\ell_2}\psi, \chi_{a_1\ell_1}; \pi)$ in (3.12). Letting N', with $N'|N$, denote the conductor of ψ, this leads to

$$\begin{aligned}
&Z(3/2 - s - w, w + 2s - 1; \chi_{a_2\ell_2}\psi, \chi_{a_1\ell_1}; \pi) \quad\quad\quad (3.13)\\
&= (N'\delta_3(a_2\ell_2\psi))^{3/2-2s-w} \sum_{n \geq 1} \frac{G_{a_2\ell_2,n}(2 - 2s - w)}{G_{a_2\ell_2,n}(w + 2s - 1)}\\
&\quad\times \frac{L(2 - 2s - w, \tilde{\chi}_n\chi_{a_2\ell_2}\overline{\psi})\chi_{a_1'\ell_1'}(n)\psi'(n)c(n)\tau(\psi)}{n^s}.
\end{aligned}$$

Here $\tau(\psi)$ is a (normalized to have absolute value 1) Gauss sum corresponding to ψ, ψ' is a new character with conductor dividing N arising from the factorization of the Gauss sum $\tau(\tilde{\chi}_n\chi_{a_2\ell_2}\psi)$ and $\chi_{a_1'\ell_1'}$ is a possibly new quadratic character modulo 8. If $\psi^2 = 1$, i.e if ψ is trivial or quadratic, then $\psi' = 1$. It is here that the case $\psi^2 = 1$ begins to diverge from the case $\psi^2 \neq 1$.

Another sieving modulo 8 is now necessary to interchange the order of summation on the right hand side of (3.13), transforming the right hand side into a linear combination of terms of the form

$$\begin{aligned}
&(N'\delta_3(a_2\ell_2\psi))^{3/2-2s-w} \sum_{n \geq 1} \frac{G_{a_2\ell_2,n}(2 - 2s - w)}{G_{a_2\ell_2,n}(w + 2s - 1)} \quad\quad (3.14)\\
&\quad\times Z(s, 2 - 2s - w; \chi_{a_2\ell_2}\psi, \chi_{a_1\ell_1}\psi'; \pi)
\end{aligned}$$

The convex hull of $R_3 \cup \beta(R_3)$ is now, in the case $r = 2$, all of $\mathbb{C}^2$. Thus after canceling new potential polar lines we have the holomorphic

continuation of $P_2(s,w)Z(s, 2-2s-w; \chi_{a_2\ell_2}, \chi_{a_1\ell_1}; \pi)$ to all of $\mathbb{C}^2$, where

$$P_2(s,w) = (s(1-s)(s+w-1/2)(3/2-s-w))^{p_1}$$
$$\times\ (w(1-w)(w+2s-1(2-2s-w))^{p_2}.$$

In the case $r = 3$, as when $r = 2$, an additional application of the β involution is applied to $Z(3/2 - s - w, 3s + 2w - 2; \chi_{a_2\ell_2}\psi, \chi_{a_1\ell_1}; \pi)$ in (3.12). This leads to

$$Z(3/2 - s - w, 3s + 2w - 2; \chi_{a_2\ell_2}\psi, \chi_{a_1\ell_1}; \pi)$$
$$= (N'\delta_3(a_2\ell_2\psi))^{5/2-3s-2w} \sum_{n\geq 1} \frac{G_{a_2\ell_2,n}(3-3s-2w)}{G_{a_2\ell_2,n}(3s+2w-2)} \qquad (3.15)$$
$$\times\ \frac{L(3-3s-2w, \tilde{\chi}_n\chi_{a_2\ell_2}\overline{\psi})\chi_{a_1'\ell_1'}(n)\psi'(n)c(n)\tau(\psi)}{n^{w+2s-1}}.$$

Here $\tau(\psi), \psi', N'$ and $\chi_{a_1'\ell_1'}$ are as in the case $r = 2$. As in that case, $\psi' = 1$ whenever ψ is trivial or quadratic.

Another sieving modulo 8 is now necessary, as when $r = 2$, to interchange the order of summation on the right hand side of (3.13), transforming the right hand side into a linear combination of terms of the form

$$(N'\delta_3(a_2\ell_2\psi))^{5/2-3s-2w} \sum_{n\geq 1} \frac{G_{a_2\ell_2,n}(3-3s-2w)}{G_{a_2\ell_2,n}(3s+2w-2)} \qquad (3.16)$$
$$\times\ Z(w+2s-1, 3-3s-2w; \chi_{a_2\ell_2}\psi, \chi_{a_1\ell_1}\psi'; \pi).$$

We now apply the α involution to $Z(w+2s-1, 3-3s-2w; \chi_{a_2\ell_2}\psi, \chi_{a_1\ell_1}\psi'; \pi)$, obtaining

$$Z(w + 2s - 1, 3 - 3s - 2w; \chi_{a_2\ell_2}\psi, \chi_{a_1\ell_1}\psi'; \pi)$$
$$= (N''\delta_4(a_1\ell_1\psi'))^{3/2-2s-w} \frac{G_\pi^{(a_1\ell_1)}(2-2s-w)}{G_\pi^{(a_1\ell_1)}(w+2s-1)} \qquad (3.17)$$
$$\times\ Z(2 - 2s - w, 3s + w - 3/2; \chi_{a_2\ell_2}\psi'', \chi_{a_1\ell_1}\overline{\psi'}; \pi).$$

Here N'' is the conductor of $\pi \otimes \psi'$, ψ'' is the new functional equation character twist of $\pi \otimes \psi'$ multiplied by ψ, and $\delta_4(a_1\ell_1\psi')$ keeps track of the extra powers of 2 introduced by the twisting. We refer to the analysis of §2.2 for the computation of N'' in the general case. However, if $\psi^2 = 1$ then $\psi' = 1$ and $\psi'' = \psi^2 = 1$. In this case ψ'' is an imprimitive identity character modulo N' and $N'' = N$.

We now need to apply the involution β one more time, to $Z(2 - 2s - w, 3s + w - 3/2; \chi_{a_2\ell_2}\psi'', \chi_{a_1\ell_1}\overline{\psi'}; \pi)$. As a consequence the GL(1)

L-series in the $3s + w - 3/2$ variable will be missing Euler factors at primes dividing N', the conductor of ψ. For this reason we write

$$
\begin{aligned}
&Z(2 - 2s - w, 3s + w - 3/2; \chi_{a_2\ell_2}\psi'', \chi_{a_1\ell_1}\overline{\psi'}; \pi) \\
&= \sum_{n \geq 1} \frac{L(3s + w - 3/2, \tilde{\chi}_n\chi_{a_2\ell_2}\psi_0'')\chi_{a_1\ell_1}(n)c(n)}{n^{2-2s-w}} \\
&\quad \times \prod_{p|N'} \left(1 - \tilde{\chi}_n\chi_{a_2\ell_2}\psi_0''(p)p^{3/2-3s-w}\right),
\end{aligned}
\tag{3.18}
$$

where ψ_0'' is the corresponding primitive character.

Applying β, and then sieving modulo 8, we end up with a linear combination of terms of the form

$$
(N'''\delta_5(a_2\ell_2\psi_0''))^{2-3s-w}(N'''')^{3/2-3s-w} \sum_{n \geq 1} \frac{G_{a_2\ell_2,n}(5/2 - 3s - w)}{G_{a_2\ell_2,n}(w + 3s - 3/2)}
$$

$$
\times \quad Z(s, 5/2 - 3s - w; \chi_{a_2\ell_2}\overline{\psi_0''}, \chi_{a_1\ell_1}\psi'; \pi).
\tag{3.19}
$$

Here N''' is the conductor of the GL(1) L-series of the character ψ' (so $N''' = 1$ if $\psi' = 1$) and N'''' is the conductor of the imprimitive part of ψ''.

We have now continued to a region whose convex closure is $\mathbb{C}^2$. Canceling the potential polar lines, we finally have, in the case $r = 3$, a holomorphic continuation of $P_3(s, w)Z(s, w; \chi_{a_2\ell_2}, \chi_{a_1\ell_1}; \pi)$ to all of $\mathbb{C}^2$, where

$$
\begin{aligned}
P_3(s, w) &= (s(1 - s)(s + w - 1/2)(3/2 - s - w)(2s + w - 1))^{p_1} \\
&\quad \times (2 - 2s - w)^{p_1}(w(1 - w)(3s + 2w - 2)(3 - 3s - 2w))^{p_2}.
\end{aligned}
$$

The above calculations are all in the literature, but included here for the reader's convenience, and for the exact dependence on the various occurrences of N, N', ..., N''''.

3.3. Collecting the analytic information.

Setting $s = 1/2$ in the information gathered above enables us to give a precise description of the analytic behavior of the functions $Z(1/2, w; \chi_{a_2\ell_2}, \chi_{a_1\ell_1}; \pi)$ as follows

Proposition 3.20. *The function*

$$
Z(1/2, w; \chi_{a_2\ell_2}, \chi_{a_1\ell_1}; \pi) = \sum_{d \geq 1} \frac{L(1/2, \pi \otimes \chi_d\chi_{a_1\ell_1})\chi_{a_2\ell_2}(d)}{d^w}
$$

converges absolutely when $\mathfrak{Re}w > 1$ and has a meromorphic continuation to $\mathbb{C}$. It has poles at

(1) $w = 1$, when $r = 1$,
(2) $w = 1$, when $r = 2$,
(3) $w = 1$ and $w = 3/4$ when $r = 3$.

Note that the function "completed" with Gamma factors (see (3.27) – (3.29)) also has poles at $w = 0$, and on GL(3) at $w = 1/4$. Away from these poles it has polynomial growth in vertical strips.

Finally, for any $a_1\ell_1, a_2\ell_2 \in \{\pm 1, \pm 2\}$, it satisfies the following functional equations. The functions C_1, C_2, and C_3 occurring below are absolute constants depending on values in $\{\pm 1, \pm 2\}$. When $r = 1$,

$$Z(1/2, w; \chi_{a_2\ell_2}, \chi_{a_1\ell_1}; \pi) \tag{3.21}$$

$$= \sum_{a_1'\ell_1', a_2'\ell_2'} C_1(a_1\ell_1, a_2\ell_2, a_1'\ell_1', a_2'\ell_2')(\delta_1(a_2'\ell_2'))^{1/2-w} \frac{G_{a_2'\ell_2'}(1-w)}{G_{a_2'\ell_2'}(w)}$$

$$\times \quad (N\delta_2(a_1'\ell_1'))^{1/2-w} \frac{G_\pi^{(a_1'\ell_1')}(1-w)}{G_\pi^{(a_1'\ell_1')}(w)} Z(1-w, s; \chi_{a_2'\ell_2'}\psi, \chi_{a_1'\ell_1'}; \pi).$$

When $r = 2$,

$$Z(1/2, w; \chi_{a_2\ell_2}, \chi_{a_1\ell_1}; \pi) \tag{3.22}$$

$$= \sum_{a_1'\ell_1', a_2'\ell_2'} C_2(a_1\ell_1, a_2\ell_2, a_1'\ell_1', a_2'\ell_2')(\delta_1(a_2'\ell_2'))^{1/2-w} \frac{G_{a_2'\ell_2'}(1-w)}{G_{a_2'\ell_2'}(w)}$$

$$\times \quad (N\delta_2(a_1'\ell_1'))^{1/2-w} \frac{G_\pi^{(a_1'\ell_1')}(1-w)}{G_\pi^{(a_1'\ell_1')}(w)} (N'\delta_3(a_2'\ell_2'\psi))^{1/2-w} \frac{G_{a_2'\ell_2'}(1-w)}{G_{a_2'\ell_2'}(w)}$$

$$\times \quad Z(1/2, 1-w; \chi_{a_2'\ell_2'}\psi, \chi_{a_1'\ell_1'}\psi'; \pi).$$

When $r = 3$,

$$Z(1/2, w; \chi_{a_2\ell_2}, \chi_{a_1\ell_1}; \pi) \tag{3.23}$$

$$= \sum_{a_1'\ell_1', a_2'\ell_2'} C_3(a_1\ell_1, a_2\ell_2, a_1'\ell_1', a_2'\ell_2')(\delta_1(a_2'\ell_2'))^{1/2-w} \frac{G_{a_2'\ell_2'}(1-w)}{G_{a_2'\ell_2'}(w)}$$

$$\times \quad (N\delta_2(a_1'\ell_1'))^{1/2-w} \frac{G_\pi^{(a_1'\ell_1')}(1-w)}{G_\pi^{(a_1'\ell_1')}(w)} (N'\delta_3(a_2'\ell_2'\psi))^{1-2w} \frac{G_{a_2'\ell_2'}(3/2-2w)}{G_{a_2'\ell_2',}(2w-1/2)}$$

$$\times \quad (N''\delta_4(a_1'\ell_1'\psi'))^{1/2-w} \frac{G_\pi^{(a_1'\ell_1')}(1-w)}{G_\pi^{(a_1'\ell_1')}(w)} (N'''\delta_5(a_2'\ell_2'\psi_0''))^{1/2-w}(N'''')^{-w}$$

$$\times \quad \frac{G_{a_2'\ell_2'}(1-w)}{G_{a_2'\ell_2'}(w)} Z(1/2, 1-w; \chi_{a_2'\ell_2'}\overline{\psi_0''}, \chi_{a_1'\ell_1'}\psi'; \pi).$$

For $w = -\epsilon + it$, with $\epsilon > 0$, lines (3.21) – (3.23) relate $Z(1/2, -\epsilon + it; \chi_{a_2\ell_2}, \chi_{a_1\ell_1}; \pi)$ to $Z(1/2, 1 + \epsilon - it; \chi_{a_2\ell_2}, \chi_{a_1\ell_1}; \pi)$. Suppose that

$$Z(1/2, 1 + \epsilon - it; \chi_{a_2\ell_2}, \chi_{a_1\ell_1}; \pi) \ll_{t,\mathfrak{q}} N^{\alpha_r},$$

with some $\alpha \geq 0$, and the implied constant depending at most polynomially on t. Then by convexity it follows that for $r = 1$,

$$Z(1/2, 1/2 + it; \chi_{a_2\ell_2}, \chi_{a_1\ell_1}; \pi) \ll N^{1/4+\alpha_1+\epsilon}, \qquad (3.24)$$

for $r = 2$,

$$Z(1/2, 1/2 + it; \chi_{a_2\ell_2}, \chi_{a_1\ell_1}; \pi) \ll (NN')^{1/4+\alpha_2+\epsilon}, \qquad (3.25)$$

and for $r = 3$,

$$Z(1/2, 1/2 + it; \chi_{a_2\ell_2}, \chi_{a_1\ell_1}; \pi) \ll (N(N')^2 N'' N''')^{1/4+\alpha_3+\epsilon}, \qquad (3.26)$$

with the implied constant including a polynomial power of t.

Here N is the conductor of π, N' is the conductor of ψ, N'' is the conductor of $\pi \otimes \psi'$, and N''' is discussed just after (3.19). If $\psi^2 = 1$ then $N'' = N$ and $N''' = 1$.

The previous proposition is in a useful form for measuring the precise growth in vertical strips via an application of Stirling's formula to the ratios of gamma factors. After multiplying by the gamma factors in the denominator, another more symmetrical formulation of the functional equation can be found. This is the form that we will use in the proof of Theorem 1.13.

The gamma factors $G_{a_2'\ell_2'}(w), G_{a_2'\ell_2',n}(w)$ are all of the form

$$G_+(w) = \pi^{-w/2}\Gamma(w/2) \text{ or } G_-(w) = (2\pi)^{-w/2}\Gamma((w+1)/2).$$

Similarly, the gamma factors $G_\pi^{(a_1'\ell_1')}(w)$ are all of the form

$$G_{+,\pi}(w) = \prod_{i=1}^{r} \pi^{-w/2}\Gamma((w+\kappa_i)/2) \text{ or } G_{-,\pi}(w) = \prod_{i=1}^{r}(2\pi)^{-w/2}\Gamma((w+\kappa_i+1)/2).$$

Consider first the case $r = 1$. After multiplying by $G_+(w)G_{+,\pi}(w)$, each piece in the summation on the right hand side of (3.21) takes the form a constant times

$$(N2^a\pi^b)^{1/2-w}\rho(w)\rho_\pi(w)G_+(1-w)G_{+,\pi}(1-w)Z(1-w, s; \chi_{a_2'\ell_2'}\psi, \chi_{a_1'\ell_1'}; \pi),$$

where

$$\rho(w) = 1 \text{ or } G_+(w)/G_-(w)$$

and

$$\rho_\pi(w) = 1 \text{ or } G_{+,\pi}(w)/G_{-,\pi}(w).$$

Consequently (3.21) can be rewritten as

$$G_+(w)G_{+,\pi}(w)Z(1/2, w; \chi_{a_2\ell_2}, \chi_{a_1\ell_1}; \pi) \tag{3.27}$$
$$= N^{1/2-w} \sum_{a_1'\ell_1', a_2'\ell_2'} (2^a\pi^b)^{1/2-w} C_1 \; \rho(w)\rho_\pi(w)$$
$$\times \; G_+(1-w)G_{+,\pi}(1-w)Z(1-w, s; \chi_{a_2'\ell_2'}\psi, \chi_{a_1'\ell_1'}; \pi).$$

Here each $a, b, C_1, \rho, \rho_\pi$ is a function of $a_1\ell_1, a_2\ell_2$ and $a_1'\ell_1', a_2'\ell_2'$.

Similarly for $r = 2$, (3.22) can be rewritten as

$$G_+(w)^2 G_{+,\pi}(w)Z(1/2, w; \chi_{a_2\ell_2}, \chi_{a_1\ell_1}; \pi) \tag{3.28}$$
$$= (NN')^{1/2-w} \sum_{a_1'\ell_1', a_2'\ell_2'} (2^a\pi^b)^{1/2-w} C_2 \; \rho(w)^2\rho_\pi(w)$$
$$\times \; G_+(1-w)^2 G_{+,\pi}(1-w)Z(s, 1-w; \chi_{a_2'\ell_2'}\psi, \chi_{a_1'\ell_1'}; \pi),$$

and for $r = 3$, (3.23) can be rewritten as

$$G_+(w)^2 G_+(2w-1/2)G_{+,\pi}(w)^2 Z(1/2, w; \chi_{a_2\ell_2}, \chi_{a_1\ell_1}; \pi) \tag{3.29}$$
$$= (NN'N''N''')^{1/2-w}(N'''')^{-w} \sum_{a_1'\ell_1', a_2'\ell_2'} (2^a\pi^b)^{1/2-w} C_3 \; \rho(w)^2\rho(2w-1/2)\rho_\pi(w)^2$$
$$\times \; G_+(1-w)^2 G_+(3/2-2w)G_{+,\pi}(1-w)^2 Z(s, 1-w; \chi_{a_2'\ell_2'}\psi, \chi_{a_1'\ell_1'}; \pi).$$

3.4. Residual Behavior.

The nature of the residues at the simple poles and leading coefficients of the Laurent expansion at higher order poles is rather complicated and varied, and clearly necessary for our application of Tauberian arguments. Fortunately, the literature already contains sufficient information in each case occurring here. Below, we collect the relevant facts.

For s in a neighborhood of $1/2$ but $s \neq 1/2$, the double Dirichlet series $Z(s, w; \chi_{a_2\ell_2}, \chi_{a_1\ell_1}; \pi)$ is holomorphic at $w = 1$ as long as $a_2\ell_2 \neq 1$. There are several polar lines intersecting the point $(1/2, 1)$. The combination of these polar lines can create multiple poles or eliminate the pole at $(1/2, 1)$, depending upon the number theoretic nature of π.

For example in the case $r = 2$ and π a cuspidal newform these lines are $w = 1$ and $w + s - 1/2 = 1$. Each pole is simple, with residues a non-zero multiple of $L(2s, \pi, \mathrm{sym}^2)$ and $L(1 - 2s, \pi, \mathrm{sym}^2)$ respectively. The residue of $Z(1/2, w; 1, \chi_{a_1\ell_1}; \pi)$ at $w = 1$ is the sum of the limits of these two residues. This sum is easily computed via the global root number, and all cases it is zero if and only if the root number is -1. This is the basis of non-vanishing theorems for families of quadratic L-series: in any case where the root number is not -1 for all twists, the two residue terms do not cancel, forcing a pole which in turn forces

the nonvanishing of infinitely many quadratic twists of the relevant L-series at $1/2$. For a full discussion of this in the case $r = 2$ see line (1.4) of [FH95] and the analysis following it.

In the case $r = 3$ and π cuspidal, if $a_2\ell_2 = 1$ and s is in a neighborhood of $1/2$ but $s \neq 1/2$, then the double Dirichlet series $Z(s, w; 1, \chi_{a_1\ell_1}; \pi)$ has a simple pole at $w = 1$ of residue a constant multiple of

$$L(2s, \pi, \mathrm{sym}^2)\zeta(6s - 1).$$

If π is itself the adjoint square lift of a cuspidal newform on GL(2), then $L(2s, \pi, \mathrm{sym}^2)$ has a pole at $s = 1/2$, and hence $Z(1/2, w; 1, \chi_{a_1\ell_1}; \pi)$ has a double pole at $w = 1$. See the discussion after Proposition 3.7 in [BFH04] for a full analysis of this case.

In all cases of interest the existence or non-existence of a pole of $Z(1/2, w; 1, \chi_{a_1\ell_1}; \pi)$ at $w = 1$ can be checked. In GL(1), GL(2) it has been verified that there is no pole at $w = 1$ if and only if the root number of every twist of π by a quadratic character is -1. On GL(3), if π is the adjoint square lift of a cuspidal newform on GL(2) there is always a pole of order 2. For generic π on GL(3) the corresponding property can be easily checked. Consequently we state the following

Proposition 3.30. *Suppose that $Z(1/2, w; 1, \chi_{a_1\ell_1}; \pi)$ has a pole of order at least 1 at $w = 1$. Denoting the coefficient of the leading term in the Laurant expression as κ, there exists some $A > 0$ such that*

$$\kappa \gg (\mathfrak{q}N)^{-A}. \tag{3.31}$$

If the L-series $L(s, \pi, \mathrm{sym}^2)$ does not have a Siegel zero (see below for the definition) then the stronger result

$$\kappa \gg (\mathfrak{q}N)^{-\epsilon} \tag{3.32}$$

is true.

Proof. If the pole is simple, then as discussed above, and also in the case $r = 1$, κ is equal to a non-zero multiple of $L(1, \pi, \mathrm{sym}^2)$. In the case of a multiple pole, the relevant value is a lower rank L-series evaluated at 1. It is well known that

$$\kappa \gg N^{-\epsilon}(1 - \beta)$$

for any β satisfying

$$1 - (1 - \log(\mathfrak{q}N))^{-1} < \beta < 1$$

and $L(s, \pi, \mathrm{sym}^2) \neq 0$ for s real with $\beta < s < 1$. Such a β is, if it exists, known as a Siegel zero. Ordinarily, finding a lower bound for $1 - \beta$ would be a subtle matter, and to obtain more refined results, it must be addressed. But in this case, all we need is a polynomial

lower bound, and hence even the weakest results in the cases $r = 1, 2$ suffice. Furthermore, the uniform bound in [Bru06] gives the GL(3) claim, as the symmetric square L-function is a factor of the Rankin-Selberg convolution $L(2s, \pi \times \pi)$ and the exterior L-series factor is easily estimated from above. $\square$

4. PROOFS OF THEOREMS 1.13 AND 1.14

Let π_r be an automorphic representation on GL(r), $r = 1, 2, 3$ of level N and let $\tilde{\pi}_r$ be its contragredient. Suppose that there exists at least one quadratic character χ such that the root number of $\pi_r \otimes \chi$ is not equal to -1.

Let $G_r(w)$ denote the gamma factors which appear in lines (3.27) – (3.29). Thus

$$G_1(w) = G_+(w)G_{+,\pi}(w),$$

$$G_2(w) = G_+(w)^2 G_{+,\pi}(w)$$

and

$$G_3(w) = G_+(w)^2 G_+(2w - 1/2)G_{+,\pi}(w)^2.$$

For $X \geq 1$, apply an inverse Mellin transform, obtaining

$$\mathcal{I} = \frac{1}{2\pi i} \int_{(2)} G_r(w) Z(1/2, w; 1, 1; \pi_r) X^w dw = \sum_d L(1/2, \pi \otimes \chi_d) V\left(\frac{d}{X}\right),$$
(4.1)

where

$$V(y) := \frac{1}{2\pi i} \int_{(2)} G_r(w) y^{-w} dw$$

has arbitrary polynomial decay for $y \gg 1$.

Move the line of integration in (4.1) to $\mathfrak{Re}\, w = -1$, passing through poles at $w = 1$, $w = 0$, and on GL(3), potential poles at $w = 3/4$ and $w = 1/4$. Depending on r, apply the functional equations (3.27) – (3.29), and make the change of variables $w \mapsto 1 - w$. The moved integral reflects into the region of absolute convergence, and breaks into a linear combination of L-series $L(1/2, \pi \otimes \chi_d)$ times new damping functions $\tilde{V}(dX/N^{2\theta_r})$. The power θ_r of N in the above is determined by the factors N, N', ..., N'''' in equations (3.27) – (3.29). For simplicity, assume that ψ has conductor N, and moreover that if $r = 3$, then ψ is quadratic. Then the power of N is $\theta_1 = 1/2$, $\theta_2 = 1$, and $\theta_3 = 2$.

Let κ_1, $\kappa_{3/4}$, $\kappa_{1/4}$, and κ_0 denote the residual contributions at $w = 1$, $3/4$, $1/4$, and 0, respectively. By Proposition 3.30, the leading term is of the order $\gg XN^{-A}$ with a possible lower order residual terms

$X^{3/4}N^B$, $X^{1/4}N^C$ and N^D for some A, B, C, and D. Let $\mathcal{R}$ denote the residual contribution. By perturbing X near N^{θ_r}, we can ensure that

$$\mathcal{R} \gg N^{-A}$$

for some fixed $A > 0$. Fix this value of X. If there is no Siegel zero, then in fact

$$\mathcal{R} \gg N^{-\epsilon}X.$$

Assume now that for all $|d| \ll N^{\theta_r+\varepsilon}$, the twisted L-series $L(1/2, \pi \otimes \chi_d)$ all vanish. As V and $\tilde{V}$ have arbitrary polynomial decay, one obtains the desired contradiction, since the residual terms are $\gg N^{-A}$ for some A. This contradiction completes the proof of Theorem 1.13.

For Theorem 1.14 in the eigenvalue aspect, one can easily apply Stirling's formula to the gamma factors corresponding to the level and complete the proof using the same argument.

Remark 4.2. Were one to attempt an improvement along the lines of §1.3, one could proceed as follows. Instead of (4.1), let $h(y)$ denote a smooth function of compact support in $[1, 2]$, say, and let $H(w)$ denote its Mellin transform, $H(w) = \int_0^\infty h(y)y^w dy/y$. The latter has arbitrary polynomial decay in vertical strips if $\mathfrak{Re}(w) > 0$. Take an integral like (4.1), except with H replacing the Gamma factors:

$$\mathcal{I} := \frac{1}{2\pi i} \int_{(2)} H(w)Z(1/2, w; 1, 1; \pi)X^w dw = \sum_{d \geq 1} L(1/2, \pi \otimes \chi_d)h(d/X).$$

Now move the contour back to $\mathfrak{Re}(w) = 1/2$, passing through the poles at $w = 1$ (and possibly at $w = 3/4$ on GL(3)). Let κ denote the residual contribution from the pole at $w = 1$, κ' the potential contribution from $w = 3/4$.

Assuming there is no Siegel zero (which has been established for our purposes in all but the classical case), then again

$$\kappa \gg N^{-\epsilon}.$$

Then inputting the supposed bound (1.23) and estimating away the residual contribution gives

$$\sum_d L(1/2, \pi \otimes \chi_d) \, h\left(\frac{d}{X}\right) \gg N^{-\varepsilon}X + \mathcal{O}\left(X^{1/2}N^\epsilon\right).$$

Take $X = N^{4\varepsilon}$. Assuming that $L(1/2, \pi \otimes \chi_d) = 0$ for all $d \leq 2X$, one obtains the desired contradiction. This completes our analysis.

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